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Coexistence theorem of steady states for nonlinear self-cross diffusion systems with competitive dynamics

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Abstract

In this paper, we discuss the existence of positive solutions to certain nonlinear elliptic systems representing competitive interaction with self-cross diffusions between two species. The method employed is the fixed point index theory in a positive cone. Sufficient conditions for the existence of positive solutions are provided.

1. Introduction

In this paper, we consider the steady state to the following strongly-coupled parabolic system:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta[\varphi(u,v)u] = uf(u,v), & \\ \frac{\partial v}{\partial t} - \Delta[\psi(u,v)v] = vg(u,v), & \text{in } \Omega \times [0,T), \\ \kappa_1 \frac{\partial u}{\partial n} + \tau_1 u = 0, & \\ \kappa_2 \frac{\partial v}{\partial n} + \tau_2 v = 0, & \text{on } \partial\Omega \times [0,T),\ T \in (0,\infty), \\ u(x,0) = \tilde{u}_0(x), \quad v(x,0) = \tilde{v}_0(x), & \text{in } \overline{\Omega}, \end{cases}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, the nonlinear functions φ, ψ, f, g satisfy certain conditions and κ_i, τ_i are nonnegative constants such that $\kappa_i^2 + \tau_i^2 \neq 0$ for $i = 1, 2$. Here Δ is the Laplacian operator and u, v may represent the densities of two competing species in many applications, namely, biology, biochemistry, ecology, immunology, etc. The functions f and g are called the *relative growth rates* of those populations. In biological interactions, two species *compete* each other if these two functions f and g are decreasing with respect to the other component, respectively. We say that the system is called *self-cross diffusion system* if the diffusions are affected by the densities of both species simultaneously.

Our research is to investigate the existence of positive solutions to the elliptic competing interacting system with self-cross diffusions:

$$\begin{cases} -\Delta[\varphi(u,v)u] = uf(u,v), \\ -\Delta[\psi(u,v)v] = vg(u,v), & \text{in } \Omega, \\ \kappa_1 \frac{\partial u}{\partial n} + \tau_1 u = 0, \\ \kappa_2 \frac{\partial v}{\partial n} + \tau_2 v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We say that system (1.1) has a positive solution (u, v) if $u(x) > 0$ and $v(x) > 0$ for all $x \in \Omega$. The existence of a positive solution (u, v) to system (1.1) is called a *positive coexistence*.

Motivated by various biological–chemical interacting models, many authors considered system (1.1) under various boundary conditions with the constant diffusion rates, i.e., $\varphi \equiv \psi \equiv$ constants. (See, for example, [3–5,7,10,12,16–18,20,23,27] and references therein.)

In recent years, there has been a considerable amount of interest to the following model with linear diffusions and growth rates:

$$\begin{cases} -\Delta[(\alpha_1 + \beta_{11}u + \beta_{12}v)u] = u(a_1 - b_{11}u - b_{12}v), \\ -\Delta[(\alpha_2 + \beta_{21}u + \beta_{22}v)v] = v(a_2 - b_{21}u - b_{22}v), & \text{in } \Omega. \end{cases} \tag{1.2}$$

System (1.2) was proposed first by Shigesada et al. in [28]. The idea is that the main reason of dispersal of two competing species is population pressures due to the mutual interference between the individuals. For a one-dimensional domain, there are several works relating to the existence of nonconstant solutions to system (1.2) under homogeneous Neumann boundary conditions [21,22]. They showed that nonconstant positive solutions exist when $\alpha_2, \beta_{21}, \beta_{22}$ are sufficiently small. In [30], system (1.2) was considered under homogeneous Dirichlet boundary conditions using the singular perturbation. He found positive solutions when certain parameters are sufficiently small. For an n-dimensional domain, in [26], Ruan studied the existence of positive solutions to the coupled competition elliptic system (1.2) with homogeneous Dirichlet boundary conditions using the index theory. Furthermore, he gave the result that the system has positive solutions when β_{12} and β_{21} are sufficiently large. In [19], Lou and Ni investigated the existence of nonconstant solutions of system (1.2) under homogeneous Neumann boundary conditions employing the method of Lyapunov functional and degree theory. For more references to the elliptic system (1.2) one can see [2,13,15,25] and references therein. Also refer to [9,14,24,32] for the corresponding parabolic system to (1.2).

In this paper, we give sufficient conditions for the existence of positive solutions of system (1.1) with competitive interactions by using the method of the fixed point index of compact operators in a positive cone. In fact, we show that if the signs of the first eigenvalues of suitable operators are both positive or both negative, or both equal to zero, then system (1.1) has a positive solution. Furthermore, our results imply that the positive steady-state solutions to the model slightly modified from (1.2) exist if some coefficients in the diffusion rates are sufficiently large when the diffusions and the growth rates are nonlinear with respect to the densities of populations. (See Section 5.) Thus ours generalizes the previous results for the competition model that φ, ψ, f, and g are linear with respect to the densities, so that one can apply our results to various biological interaction models.

This article is organized as follows. In Section 2, we state some known lemmas and give the existence and uniqueness theorem of positive solutions to a certain scalar equation. In Section 3, we give an a priori bound for nonnegative solutions of (1.1) and state the existence theorem of positive solutions to the coupled nonlinear elliptic system (1.1). In Section 4, we prove the existence theorem in Section 3 by using the index theory. Finally, we provide some remarks and consider some special cases to our system (1.1) in Section 5.

2. Preliminaries

In this section, we consider a certain eigenvalue problem which is useful throughout this paper and give some known results for fixed point index theory. The existence theorem for a scalar equation is also provided.

2.1. Certain eigenvalue problem

For $a(x) > 0$ in $C^2(\overline{\Omega})$ and $b(x) \in L^\infty(\Omega)$, consider the eigenvalue problem

$$\begin{cases} \Delta[a(x)u] + b(x)u = \lambda u, & \text{in } \Omega, \\ \kappa \frac{\partial u}{\partial n} + \tau u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where κ and τ are nonnegative constants such that $\kappa^2 + \tau^2 \neq 0$. If we define the operator $Lu := a(x)(\Delta[a(x)u] + b(x)u)$ under the boundary condition $\kappa(\partial u/\partial n) + \tau u = 0$ on $\partial\Omega$, then the formal adjoint operator of L becomes $L^*v = a(x)(\Delta[a(x)v] + b(x)v)$ and $\kappa(\partial v/\partial n) + \tau v = 0$ on $\partial\Omega$, and thus L is formally symmetric. Define the symmetric bilinear form by $B[u, v] := \int_\Omega a(x)(\Delta[a(x)u] + b(x)u)v\,dx$. Then we have

$$B[u, v] = \begin{cases} \int_\Omega (-\nabla[a(x)u]\nabla[a(x)v] + a(x)b(x)uv) \\ \quad - \int_{\partial\Omega} a(x)\left(\frac{\tau}{\kappa}a(x) - \frac{\partial a(x)}{\partial n}\right)uv, & \text{if } \kappa \neq 0, \\ \int_\Omega (-\nabla[a(x)u]\nabla[a(x)v] + a(x)b(x)uv), & \text{if } \kappa = 0. \end{cases}$$

The argument is valid for the case of $\kappa = 0$, and so we only consider the case of $\kappa \neq 0$. Using the same argument in [8], we can obtain the eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{\phi_n\}$ of (2.1) such that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots$ and $\lim_{n\to\infty} \lambda_n = -\infty$, where $n \geqslant 1$, and we have $B[\phi_n, \phi_n] = \lambda_n \|\sqrt{a(x)}\phi_n\|_{L^2}^2$, where $\|\cdot\|_{L^2}$ denotes the usual L^2-norm in Ω. Furthermore, we can see that the eigenfunction ϕ_1 of (2.1) corresponding to the eigenvalue λ_1 is unique and positive. Define the quadratic functionals on $W^{1,2}(\Omega)$ by

$$Q(\phi) = \frac{B[\phi,\phi]}{\|\sqrt{a(x)}\phi\|_{L^2}^2}$$
$$= \frac{\int_\Omega(-|\nabla[a(x)\phi]|^2 + a(x)b(x)\phi^2) - \int_{\partial\Omega} a(x)((\tau/\kappa)a(x) - \partial a(x)/\partial n)\phi^2}{\|\sqrt{a(x)}\phi\|_{L^2}^2}.$$

Then one can easily check that $\lambda_1 = \sup_{\phi\in W^{1,2}(\Omega)} Q(\phi) = Q(\phi_1)$ by the same method in [29, Section 11.A] or [11, Chapter 6].

Throughout this paper, let $\lambda_1^{(\kappa,\tau)}(\Delta a(x) + b(x))$ denote the principal eigenvalue λ_1 of the eigenvalue problem (2.1) corresponding to the unique positive principal eigenfunction $\phi_1(x)$. The following is obtained from the variational property of the principal eigenvalues of problem (2.1).

Lemma 2.1. *Assume that $b_1(x)/a_1(x) > b_2(x)/a_2(x)$, where $a_i(x) > 0$ in $C^2(\overline{\Omega})$ and $b_i(x) \in L^\infty(\Omega)$ for $i = 1, 2$.*

(i) *If $\lambda_1^{(\kappa,\tau)}(\Delta a_1(x) + b_1(x)) \leqslant 0$, then $\lambda_1^{(\kappa,\tau)}(\Delta a_2(x) + b_2(x)) < 0$.*
(ii) *If $\lambda_1^{(\kappa,\tau)}(\Delta a_2(x) + b_2(x)) \geqslant 0$, then $\lambda_1^{(\kappa,\tau)}(\Delta a_1(x) + b_1(x)) > 0$.*

Proof. Since the proof of (ii) is virtually the same as (i), we only prove (i).

Let ϕ_1 and ϕ_2 be the principal eigenfunctions corresponding to the principal eigenvalues $\lambda_1^{(\kappa,\tau)}(\Delta a_1(x) + b_1(x))$ and $\lambda_1^{(\kappa,\tau)}(\Delta a_2(x) + b_2(x))$, respectively, and

$$B_i[\phi,\phi] = \int_\Omega \left(-\left|\nabla\left[a_i(x)\phi\right]\right|^2 + a_i(x)b_i(x)\phi^2\right) + \int_{\partial\Omega}\left(a_i(x)\phi\frac{\partial(a_i(x)\phi)}{\partial n}\right)$$

and

$$Q_i(\phi) = \frac{B_i[\phi,\phi]}{\|\sqrt{a_i(x)}\phi\|_{L^2}^2}$$

for $i = 1, 2$. Since $\lambda_1^{(\kappa,\tau)}(\Delta a_1(x) + b_1(x)) = \sup_{\phi\in W^{1,2}(\Omega)} Q_1(\phi) \leqslant 0$, $B_1[\phi,\phi] \leqslant 0$ for all $\phi \in W^{1,2}(\Omega)$. If we take $\phi := (a_2(x)/a_1(x))\phi_2$, then

$$\begin{aligned}
B_1[\phi,\phi] &= \int_\Omega\left(-\left|\nabla\left[a_2(x)\phi_2\right]\right|^2 + \frac{b_1(x)}{a_1(x)}\big(a_2(x)\big)^2\phi_2^2\right) + \int_{\partial\Omega}\left(a_2(x)\phi_2\frac{\partial(a_2(x)\phi_2)}{\partial n}\right)\\
&> \int_\Omega\left(-\left|\nabla\left[a_2(x)\phi_2\right]\right|^2 + a_2(x)b_2(x)\phi_2^2\right) + \int_{\partial\Omega}\left(a_2(x)\phi_2\frac{\partial(a_2(x)\phi_2)}{\partial n}\right)\\
&= B_2[\phi_2,\phi_2].
\end{aligned}$$

So we can conclude that

$$\lambda_1^{(\kappa,\tau)}\big(\Delta a_2(x) + b_2(x)\big) = \frac{B_2[\phi_2,\phi_2]}{\|\sqrt{a_2(x)}\phi_2\|_{L^2}^2} < \frac{B_1[\phi,\phi]}{\|\sqrt{a_2(x)}\phi_2\|_{L^2}^2} \leqslant 0. \qquad \square$$

Remark 2.2. If $(\partial/\partial n)(a(x)) \leqslant 0$, then

$$\int\limits_{\partial\Omega} \tilde{\phi}\frac{\partial\tilde{\phi}}{\partial n} = -\int\limits_{\partial\Omega} a(x)\left(\frac{\tau}{\kappa}a(x) - \frac{\partial a(x)}{\partial n}\right)\phi^2 \leqslant 0,$$

where $\tilde{\phi} := a(x)\phi$. Thus from

$$\lambda_1^{(\kappa,\tau)}\big(\Delta a(x) + b(x)\big) = \sup_{\tilde{\phi}\in W^{1,2}(\Omega)} \frac{\int_\Omega(-|\nabla\tilde{\phi}|^2 + (b(x)/a(x))\tilde{\phi}^2) + \int_{\partial\Omega}\tilde{\phi}(\partial\tilde{\phi}/\partial n)}{\|\tilde{\phi}/\sqrt{a(x)}\|_{L^2}^2}$$

we can see that $\lambda_1^{(\kappa,\tau)}(\Delta a(x) + b(x)) < 0$ for sufficiently large $a(x)$ or sufficiently small $b(x)$.

Lemma 2.3. *Let $a(x) > 0$ in $C^2(\overline{\Omega})$, $b(x) \in L^\infty(\Omega)$, and $u \geqslant 0$, $u \not\equiv 0$ in Ω with $\kappa(\partial u/\partial n) + \tau u = 0$ on $\partial\Omega$.*

(i) *If $0 \not\equiv (\Delta a(x) + b(x))u \geqslant 0$, then $\lambda_1^{(\kappa,\tau)}(\Delta a(x) + b(x)) > 0$.*
(ii) *If $0 \not\equiv (\Delta a(x) + b(x))u \leqslant 0$, then $\lambda_1^{(\kappa,\tau)}(\Delta a(x) + b(x)) < 0$.*
(iii) *If $(\Delta a(x) + b(x))u \equiv 0$, then $\lambda_1^{(\kappa,\tau)}(\Delta a(x) + b(x)) = 0$.*

Proof. We only prove (i). Let $\phi(x) > 0$ be the eigenfunction corresponding to the principal eigenvalue $\lambda_1^{(\kappa,\tau)}(\Delta a(x) + b(x))$. Then

$$0 < \int\limits_\Omega a(x)\phi\big(\Delta\big[a(x)u\big] + b(x)u\big) = \lambda_1^{(\kappa,\tau)}\big(\Delta a(x) + b(x)\big)\int\limits_\Omega a(x)\phi u.$$

Since $u \not\equiv 0$, $\lambda_1^{(\kappa,\tau)}(\Delta a(x) + b(x)) > 0$. □

The following lemma can be shown by the similar manner in [17, Lemma 2]. Let $T : E \to E$ be a linear operator on a Banach space and denote the spectral radius of T by $r(T)$.

Lemma 2.4. *Let $a(x) > 0$ in $C^2(\overline{\Omega})$, $b(x) \in L^\infty(\Omega)$, and M be a positive constant such that $b(x) + Ma(x) > 0$ for all $x \in \overline{\Omega}$. Then we have*

$$\text{(i)}\quad \lambda_1^{(\kappa,\tau)}\big(\Delta a(x) + b(x)\big) > 0 \quad\Rightarrow\quad r\left[\frac{1}{a(x)}(-\Delta + M)^{-1}\big(b(x) + Ma(x)\big)\right] > 1;$$

$$\text{(ii)}\quad \lambda_1^{(\kappa,\tau)}\big(\Delta a(x) + b(x)\big) < 0 \quad\Rightarrow\quad r\left[\frac{1}{a(x)}(-\Delta + M)^{-1}\big(b(x) + Ma(x)\big)\right] < 1;$$

$$\text{(iii)}\quad \lambda_1^{(\kappa,\tau)}\big(\Delta a(x) + b(x)\big) = 0 \quad\Rightarrow\quad r\left[\frac{1}{a(x)}(-\Delta + M)^{-1}\big(b(x) + Ma(x)\big)\right] = 1.$$

2.2. A scalar equation

In this section, we consider the scalar equation

$$\begin{cases} -\Delta[\varphi(u)u] = uf(u), & \text{in } \Omega, \\ \kappa \frac{\partial u}{\partial n} + \tau u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and κ, τ are nonnegative constants such that $\kappa^2 + \tau^2 \neq 0$. The functions $\varphi : [0,\infty) \to [0,\infty)$ and $f : [0,\infty) \to \mathbb{R}$ are assumed to satisfy the following hypotheses:

(H2.1) $\varphi(0) > 0$ and $\varphi(u)$ is C^2-function in u with $\varphi_u(u) \geqslant 0$ for all $u \geqslant 0$.
(H2.2) $f(u)$ is C^1-function in u with $f_u(u) < 0$ for all $u \geqslant 0$.
(H2.3) $f(0) > 0$ and $f(u) < 0$ on (C_0, ∞) for some positive constant C_0.

Remark 2.5. Hypothesis (H2.1) implies that the map $G_\varphi(u) := \varphi(u)u$ has a continuous inverse in u since $\partial G_\varphi/\partial u = \varphi_u(u)u + \varphi(u) > 0$ for all $u \geqslant 0$. Denote this inverse map by $G_\varphi^{-1}(u)$. For this inverse map, we can see that $(\partial/\partial u)(G_\varphi^{-1}(u)) > 0$ for all $u \geqslant 0$ by the inverse function theorem.

Definition 2.6. A function $u(x)$ is called a *solution* of (2.2) if $\varphi(u)u \in C^{2,\alpha}(\overline{\Omega})$, where $0 < \alpha < 1$ and $u(x)$ satisfies (2.2).

Definition 2.7. A function $\hat{u}(x)$ is called an *upper solution* of (2.2) if $\hat{u}$ satisfies the following conditions:

$$-\Delta\big[\varphi(\hat{u})\hat{u}\big] \geqslant \hat{u} f(\hat{u}) \quad \text{in } \Omega, \qquad \kappa \frac{\partial \hat{u}}{\partial n} + \tau \hat{u} \geqslant 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

Similarly, we define a *lower solution* of (2.2) by reversing the inequalities in (2.3).

Using the strong maximum principle, we can easily obtain the following lemma.

Lemma 2.8. *The nonnegative solution $u(x)$ of* (2.2) *with hypotheses* (H2.1)–(H2.3) *has an a priori bound*; $u(x) \leqslant C_0$ *for all* $x \in \overline{\Omega}$.

Corollary 2.9. *The nonnegative solution $u(x)$ of* (2.2) *with hypotheses* (H2.1)–(H2.3) *satisfies* $f(u(x)) \geqslant 0$.

Proof. In Lemma 2.8, take a constant C_0 as the unique root of $f(u) = 0$ in u. □

Remark 2.10. By virtue of Corollary 2.9, we can easily see that $f(u)/\varphi(u)$ is monotone decreasing with respect to u on $[0, C_0]$ where C_0 is the unique root of $f(u) = 0$ in u.

Now we give the existence and uniqueness theorem of positive solutions of (2.2).

Theorem 2.11. *Consider the scalar equation* (2.2) *with hypotheses* (H2.1)–(H2.3).

(i) *If* $\lambda_1^{(\kappa,\tau)}(\Delta\varphi(0) + f(0)) \leqslant 0$, *then* (2.2) *has no positive solutions.*
(ii) *If* $\lambda_1^{(\kappa,\tau)}(\Delta\varphi(0) + f(0)) > 0$, *then* (2.2) *has a unique positive solution.*

Proof. (i) By contraries, suppose $u(x)$ is a positive solution of (2.2). Then $\lambda_1^{(\kappa,\tau)}(\Delta\varphi(u) + f(u)) = 0$ by Lemma 2.3(iii), and so $\lambda_1^{(\kappa,\tau)}(\Delta\varphi(0) + f(0)) > 0$ by Lemma 2.1(ii) since $f(u)/\varphi(u) < f(0)/\varphi(0)$ by Remark 2.10.

(ii) Define an operator $F : [[0, \hat{u}]] \to C(\overline{\Omega})$ by $F := G_\varphi^{-1} \circ H$, where $[[0, \hat{u}]]$ denotes the ordered interval in $C_{\kappa,\tau}(\overline{\Omega}) := \{u \in C(\overline{\Omega})\colon \kappa(\partial u/\partial n) + \tau u = 0 \text{ on } \partial\Omega\}$. Here G_φ^{-1} is the continuous inverse of the map $G_\varphi(u) = \varphi(u)u$ in u which is defined in Remark 2.5 and H is given by $Hu := (-\Delta + M)^{-1}[(f(u) + M\varphi(u))u]$, where M is a positive constant large enough so that $(f(u)+M\varphi(u))u$ is monotone increasing with respect to u. Such a constant M exists by (H2.1). Notice that the operator F is a positive monotone increasing compact map. We may observe that u is a solution of (2.2) if and only if u is a fixed point of F. Let $\hat{u}(x) = C_0$, where C_0 is a positive constant in (H2.3). Then we can easily check that $\hat{u}(x)$ is an upper solution of (2.2), i.e., $-\Delta[\varphi(\hat{u})\hat{u}] \geqslant \hat{u} f(\hat{u})$ in Ω and $\kappa(\partial\hat{u}/\partial n) + \tau\hat{u} \geqslant 0$ on $\partial\Omega$. Adding $M\varphi(\hat{u})\hat{u}$ and applying $G_\varphi^{-1} \circ (-\Delta + M)^{-1}$ both sides, we have $F(\hat{u}) \leqslant \hat{u}$. Also note that $\bar{u} = 0$ is a solution of (2.2) and we have $F'(\bar{u}) = F'(0) = (1/\varphi(0))(-\Delta + M)^{-1}(f(0) + M\varphi(0))$ by the simple calculation. So by Lemma 2.4(i), the assumption $\lambda_1^{(\kappa,\tau)}(\Delta\varphi(0)+f(0)) > 0$ implies $r(F'(0)) > 1$. Now apply Theorem 7.6 in [1] to conclude that there is a positive maximal solution $u \gg 0$ in $[[0, \hat{u}]]$.

Finally we show the uniqueness of positive solutions of (2.2). Let u_{M} be the maximal solution of (2.2) and u_1 be an another positive solution of (2.2). Then we have $\lambda_1^{(\kappa,\tau)}(\Delta\varphi(u_1) + f(u_1)) = \lambda_1^{(\kappa,\tau)}(\Delta\varphi(u_{\mathrm{M}}) + f(u_{\mathrm{M}})) = 0$ by Lemma 2.3(iii). Since u_{M} is a maximal solution of (2.2), $u_1 \leqslant u_{\mathrm{M}}$. Contrariwise, if we assume that $u_1 \not\equiv u_{\mathrm{M}}$, then $f(u_{\mathrm{M}})/\varphi(u_{\mathrm{M}}) < f(u_1)/\varphi(u_1)$ by Remark 2.10. Thus $\lambda_1^{(\kappa,\tau)}(\Delta\varphi(u_{\mathrm{M}}) + f(u_{\mathrm{M}})) < 0$ by Lemma 2.1(i), which is a contradiction. This completes the proof. □

2.3. *Fixed point index theory*

Let E be a real Banach space and $W \subset E$ a closed convex set. W is called a *total wedge* if $\alpha W \subset W$ for all $\alpha \geqslant 0$ and $\overline{W - W} = E$. A wedge is said to be a *cone* if $W \cap (-W) = \{0\}$. For $y \in W$, define $W_y = \{x \in E\colon y + \gamma x \in W \text{ for some } \gamma > 0\}$ and $S_y = \{x \in \overline{W}_y\colon -x \in \overline{W}_y\}$. Then $\overline{W}_y$ is a wedge containing $W, y, -y$, while S_y is a closed subspace of E containing y. Let T be a compact linear operator on E which satisfies $T(\overline{W}_y) \subset \overline{W}_y$. We say that T has *property* α on $\overline{W}_y$ if there is $t \in (0, 1)$ and $w \in \overline{W}_y \setminus S_y$ such that $w - tTw \in S_y$. Let $F : W \to W$ is a compact operator with a fixed point $y \in W$ and F is Fréchet differentiable at y. Let $L = F'(y)$ be the Fréchet derivative of F at y. Then L maps $\overline{W}_y$ into itself. For an open subset $U \subset W$, define $\mathrm{index}_W(F, U) = \mathrm{index}(F, U, W) = \deg_W(I - F, U, 0)$, where I is the identity map. If y is an isolated fixed point of F, then the fixed point index of F at y in W is defined by $\mathrm{index}_W(F, y) = \mathrm{index}(F, y, W) = \mathrm{index}(F, U(y), W)$, where $U(y)$ is a small open neighborhood of y in W.

In [6], Dancer introduced the formula to explicitly evaluate the indices of a compact operator at the isolated fixed points on cones in a Banach space. Later, this result was improved by Li [16], Wang et al. [31] and Ruan and Feng [27]. As the authors pointed out in [27], the result [27] is equivalent to [31] in the case of $E = \overline{W_y - W_y}$. The following can be obtained from the results of [6,16,31].

Theorem 2.12. *Assume that $I - L$ is invertible on $\overline{W}_y$.*

(i) *If L has property α on $\overline{W}_y$, then* $\text{index}_W(F, y) = 0$.
(ii) *If L does not have property α on $\overline{W}_y$, then* $\text{index}_W(F, y) = (-1)^\sigma$*, where σ is the sum of multiplicities of all the eigenvalues of L which are greater than* 1.

3. Existence theorem

In this section, we give an a priori bound for the positive solution of (1.1) and state the existence theorem of positive solutions to system (1.1).

We impose the following hypotheses in system (1.1):

(H3.1) $\varphi(0,0) > 0$, $\psi(0,0) > 0$, and $\varphi(u,v), \psi(u,v)$ are C^2-functions in u, v with $\varphi_u, \varphi_v, \psi_u, \psi_v \geqslant 0$ for all $(u,v) \in [0,\infty) \times [0,\infty)$.
(H3.2) φ is concave down with respect to v and ψ is concave down with respect to u.
(H3.3) $f(u,v), g(u,v)$ are C^1-functions in u, v with $f_u, f_v, g_u, g_v < 0$ for all $(u,v) \in [0,\infty) \times [0,\infty)$.
(H3.4) There exist constants $C_1 > 0$ and $C_2 > 0$ such that $f(u,0) < 0$ on $u \in (C_1, \infty)$ and $g(0,v) < 0$ on $v \in (C_2, \infty)$.
(H3.5) There exists a constant $C_3 > 0$ such that $f(0,v) < 0$ on $v \in (C_3, \infty)$.

In the above assumptions, (H3.3) represents the competing interactions between two species, (H3.4) gives the logistic property of growth rates for each species, and (H3.5) implies that $f(0,v)$ does not exponentially decrease.

Definition 3.1. A pair of functions (u,v) is called a *solution* of (1.1) if $\varphi(u,v)u$, $\psi(u,v)v \in C^{2,\alpha}(\overline{\Omega})$, where $0 < \alpha < 1$ and (u,v) satisfies (1.1).

By Theorem 2.11(ii), if $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,0) + f(0,0)) > 0$ in addition to (H3.1)–(H3.5), then there is a nonnegative nonzero solution $(u_0, 0)$ of (1.1) where u_0 is the unique positive solution to the equation, $-\Delta[\varphi(u,0)u] = uf(u,0)$ in Ω and $\kappa_1(\partial u/\partial n) + \tau_1 u = 0$ on $\partial\Omega$. Similarly, if $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(0,0) + g(0,0)) > 0$, then there is a nonnegative nonzero solution $(0, v_0)$ of (1.1), where v_0 is the unique positive solution to the equation, $-\Delta[\psi(0,v)v] = vg(0,v)$ in Ω and $\kappa_2(\partial v/\partial n) + \tau_2 v = 0$ on $\partial\Omega$. These solutions $(u_0, 0)$ and $(0, v_0)$ are called *semitrivial* solutions and play an important role for the existence of positive solutions to system (1.1).

Lemma 3.2. *Let (u,v) be a positive solution of* (1.1). *If $\varphi(u,v)u$ and $\psi(u,v)v$ attain their maximum at $x = x_0$ and $x = x_1$ over $\overline{\Omega}$, respectively, then $f(u(x_0), v(x_0)) \geqslant 0$ and $g(u(x_1), v(x_1)) \geqslant 0$.*

Proof. We only prove $f(u(x_0), v(x_0)) \geqslant 0$ since $g(u(x_1), v(x_1)) \geqslant 0$ can be shown similarly.

If $x_0 \in \Omega$, then we have $-\Delta[\varphi(u(x_0), v(x_0))u(x_0)] \geqslant 0$, and so the result is obvious. When $x_0 \in \partial\Omega$, we have the following two cases.

Case 1: $\kappa_1 = 0$. Since $u = 0$ on $\partial\Omega$, $\varphi(u, v)u = 0$ on $\partial\Omega$, and so $\max_{x\in\overline{\Omega}}\{\varphi(u, v)u\} = 0$, but this is impossible. Thus $x_0 \notin \partial\Omega$ in this case.

Case 2: $\kappa_1 \neq 0$. If $\kappa_2 = 0$, then $v = 0$ on $\partial\Omega$, and so we have $\partial v/\partial n \leqslant 0$ on $\partial\Omega$ by the positivity of the solution v. Thus

$$\frac{\partial}{\partial n}\big(\varphi(u, v)u\big) = \left(\varphi_u \frac{\partial u}{\partial n} + \varphi_v \frac{\partial v}{\partial n}\right)u + \varphi\frac{\partial u}{\partial n} = -(\varphi_u u + \varphi)\left(\frac{\tau_1}{\kappa_1}u\right) + \varphi_v u\frac{\partial v}{\partial n} \leqslant 0 \quad \text{on } \partial\Omega.$$

Also if $\kappa_2 \neq 0$, then

$$\frac{\partial}{\partial n}\big(\varphi(u, v)u\big) = -(\varphi_u u + \varphi)\left(\frac{\tau_1}{\kappa_1}u\right) - \varphi_v u\left(\frac{\tau_2}{\kappa_2}v\right) \leqslant 0 \quad \text{on } \partial\Omega.$$

Consequently, we can see

$$\frac{\partial}{\partial n}\big(\varphi(u, v)u\big) \leqslant 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

If we assume that $f(u(x_0), v(x_0)) < 0$, then there is a small ball B such that $\partial B \cap \partial\Omega = \{x_0\}$ and $f(u, v) < 0$ for all $x \in B$. Since $\varphi(u, v)u$ has a maximum at $x = x_0$, we have $(\partial/\partial n)(\varphi(u(x_0), v(x_0))u(x_0)) > 0$ by Hopf's lemma, which is a contradiction to (3.1). □

Lemma 3.3. *Any positive solution* (u, v) *of* (1.1) *with hypotheses* (H3.1)–(H3.5) *has an a priori bound.*

Proof. Let (u, v) be a positive solution of (1.1) and $G_{\varphi_0}^{-1}(u)$ the continuous inverse of the map $G_{\varphi_0}(u) = \varphi(u, 0)u$ in u. We claim that $u(x) \leqslant G_{\varphi_0}^{-1}(\varphi(C_1, C_3)C_1)$ for all $x \in \overline{\Omega}$. To show this, assume $\varphi(u(x_0), v(x_0))u(x_0) = \max_{x\in\overline{\Omega}}\{\varphi(u, v)u\}$. Then $f(u(x_0), v(x_0)) \geqslant 0$ by Lemma 3.2. Since $f(u(x_0), 0) \geqslant f(u(x_0), v(x_0)) \geqslant 0 \geqslant f(C_1, 0)$ and $f(0, v(x_0)) \geqslant f(u(x_0), v(x_0)) \geqslant 0 \geqslant f(0, C_3)$, we can see that $u(x_0) \leqslant C_1$ and $v(x_0) \leqslant C_3$. Therefore $\max_{x\in\overline{\Omega}}\{\varphi(u, v)u\} \leqslant \varphi(C_1, C_3)C_1$, and so

$$\varphi(u, 0)u \leqslant \max_{x\in\overline{\Omega}}\big\{\varphi(u, 0)u\big\} \leqslant \max_{x\in\overline{\Omega}}\big\{\varphi(u, v)u\big\} \leqslant \varphi(C_1, C_3)C_1.$$

Finally we can conclude that $u(x) \leqslant G_{\varphi_0}^{-1}(\varphi(C_1, C_3)C_1)$ for all $x \in \overline{\Omega}$.

By the similar argument, we can show that $v(x) \leqslant G_{\psi_0}^{-1}(\psi(Q, C_2)C_2)$, where $Q := G_{\varphi_0}^{-1}(\varphi(C_1, C_3)C_1)$ and $G_{\psi_0}^{-1}(v)$ is the continuous inverse of the map $G_{\psi_0}(v) := \psi(0, v)v$ in v. □

Throughout this paper, let Q and R be a priori bounds for u and v, respectively. That is, the nonnegative solutions u and v of (1.1) satisfy $u(x) \leqslant Q$ and $v(x) \leqslant R$. Now we state the existence theorem of positive solutions to system (1.1) which will be proved in Section 4.

Theorem 3.4. *Suppose that system* (1.1) *satisfies hypotheses* (H3.1)–(H3.5)*. Assume* $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,0)+f(0,0))>0$ *and* $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(0,0)+g(0,0))>0$*. Then* (1.1) *has a positive solution provided that* $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,v_0)+f(0,v_0))$ *and* $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(u_0,0)+g(u_0,0))$ *have the same signs, i.e., the signs are both positive or both negative or both equal to* 0*.*

4. Proof of existence theorem

Define the functions $G=(w_1,w_2)$ and $\mathbf{s}=(s_1,s_2)$ by

$$w_1(u,v)=\varphi(u,v)u,\qquad w_2(u,v)=\psi(u,v)v,$$
$$s_1(u,v)=u\big(f(u,v)+M\varphi(u,v)\big),\qquad s_2(u,v)=v\big(g(u,v)+M\psi(u,v)\big),$$

where M is a sufficiently large positive constant so that s_1 is monotone increasing with respect to u and s_2 is monotone increasing with respect to v for all $(u,v)\in[0,Q]\times[0,R]$. The existence of M follows from the hypotheses $\varphi(0,0)>0$ and $\psi(0,0)>0$. Using hypotheses (H3.1) and (H3.2), we have $\partial G(u,v)/\partial(u,v)>0$ since

$$\begin{aligned}\frac{\partial(w_1,w_2)}{\partial(u,v)}&=(\varphi_u u+\varphi)(\psi_v v+\psi)-\varphi_v\psi_u uv\geqslant\varphi\psi-\varphi_v\psi_u uv\\&=uv\left[\frac{\varphi(u,v)-\varphi(u,0)+\varphi(u,0)}{v}\,\frac{\psi(u,v)-\psi(0,v)+\psi(0,v)}{u}-\varphi_v\psi_u\right]\\&=uv\left[\left(\varphi_v(u,\eta)+\frac{\varphi(u,0)}{v}\right)\left(\psi_u(\xi,v)+\frac{\psi(0,v)}{u}\right)-\varphi_v\psi_u\right]>0,\end{aligned}$$

where $0<\xi\leqslant u$, $0<\eta\leqslant v$, and $(u,v)\in(0,\infty)\times(0,\infty)$. Hence G is invertible and denote the inverse of $G(u,v)$ by $G^{-1}(u,v)$. Let us define a compact operator $H:C(\overline{\Omega})\times C(\overline{\Omega})\to C(\overline{\Omega})\times C(\overline{\Omega})$ by $H(u,v)=((-\Delta+M)^{-1}s_1(u,v),(-\Delta+M)^{-1}s_2(u,v))$. Then we may observe that (1.1) is equivalent to $(u,v)=(G^{-1}\circ H)(u,v)$. Denote $F:=G^{-1}\circ H$ throughout this section.

We introduce the following notations:

(i) $C_{\kappa_i,\tau_i}(\overline{\Omega}):=\{u\in C(\overline{\Omega})\colon\ \kappa_i(\partial u/\partial n)+\tau_i u=0 \text{ on } \partial\Omega\}$;
(ii) $E:=C_{\kappa_1,\tau_1}(\overline{\Omega})\oplus C_{\kappa_2,\tau_2}(\overline{\Omega})$;
(iii) $D:=\{(u,v)\in C_{\kappa_1,\tau_1}(\overline{\Omega})\oplus C_{\kappa_2,\tau_2}(\overline{\Omega})\colon\ u\leqslant Q+1,\ v\leqslant R+1\}$;
(iv) $K_i:=\{u\in C_{\kappa_i,\tau_i}(\overline{\Omega})\colon\ 0\leqslant u(x),\ x\in\overline{\Omega}\}$;
(v) $W:=K_1\oplus K_2$;
(vi) $P_\rho:=\{(u,v)\in W\colon\ u\leqslant\rho,\ v\leqslant\rho\}$;
(vii) $D':=(\operatorname{int}D)\cap W$ for $\rho>0$.

Note that D' is open in W and every positive solution of (1.1) is a fixed point of the compact operator F in D'. To show that system (1.1) has a strictly positive solution (u,v), we prove that F has a nontrivial fixed point in D'. So we need to calculate the fixed point index for the trivial solution $(0,0)$ and semitrivial solutions $(u_0,0)$ and $(0,v_0)$. We also require that the point be an isolated fixed point to use the fixed point index for an operator at a point. If such fixed points are not isolated, then there must be a nontrivial fixed point in the interior of D', so that the system has a positive solution. Thus we may assume that $(0,0)$, $(u_0,0)$, and $(0,v_0)$ are isolated fixed points of F and so $\mathrm{index}_W(F,(0,0))$, $\mathrm{index}_W(F,(u_0,0))$, and $\mathrm{index}_W(F,(0,v_0))$ are well defined.

The next lemma is useful in the calculation of the fixed point index.

Lemma 4.1. *Assume that hypotheses* (H3.1)–(H3.5) *are satisfied.*

(i) *If* $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,0)+f(0,0))>0$, *then* $\lambda_1^{(\kappa_1,\tau_1)}(\Delta[\varphi_u(u_0,0)u_0+\varphi(u_0,0)]+f(u_0,0)+u_0f_u(u_0,0))<0$.
(ii) *If* $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(0,0)+g(0,0))>0$, *then* $\lambda_1^{(\kappa_2,\tau_2)}(\Delta[\psi_v(0,v_0)v_0+\psi(0,v_0)]+g(0,v_0)+v_0g_v(0,v_0))<0$.

Proof. As in Corollary 2.9, we can have $f(u_0,0)\geqslant 0$ and $g(0,v_0)\geqslant 0$, and so the inequalities

$$\frac{f(u_0,0)+u_0f_u(u_0,0)}{\varphi_u(u_0,0)u_0+\varphi(u_0,0)}<\frac{f(u_0,0)}{\varphi(u_0,0)}\quad\text{and}\quad\frac{g(0,v_0)+v_0g_v(0,v_0)}{\psi_u(0,v_0)v_0+\psi(0,v_0)}<\frac{g(0,v_0)}{\psi(0,v_0)}$$

hold. Since $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(u_0,0)+f(u_0,0))=\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(0,v_0)+g(0,v_0))=0$ by Lemma 2.3(iii), the desired results follow from Lemma 2.1(i). $\square$

We now calculate the index of F at each trivial and semitrivial steady-state solutions under the different signs of the first eigenvalues of the operators $\Delta\varphi(0,v_0)+f(0,v_0)$ and $\Delta\psi(u_0,0)+g(u_0,0)$. In the following Lemmas 4.2–4.5, we assume that $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,0)+f(0,0))>0$ and $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(0,0)+g(0,0))>0$, so that there exist semitrivial solutions u_0 and v_0.

Lemma 4.2. $\mathrm{index}_W(F,(0,0))=0$.

Proof. Let $\rho=\max\{Q,R\}+1$. Observe that $F(0,0)=(0,0)$ and F is compact on P_ρ. Let $L:=F'(0,0)$, where $F'(0,0)$ is the Fréchet derivative of F at $(0,0)$. Then by the calculation, we have

$$L\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}\frac{1}{\varphi(0,0)}(-\Delta+M)^{-1}[(f(0,0)+M\varphi(0,0))\xi]\\ \frac{1}{\psi(0,0)}(-\Delta+M)^{-1}[(g(0,0)+M\psi(0,0))\eta]\end{pmatrix}$$

for each $(\xi,\eta)\in E$.

First we show that 1 is not an eigenvalue of L corresponding to a positive eigenfunction $\binom{\xi}{\eta}$. Assume that L has an eigenvalue 1, i.e., $L\binom{\xi}{\eta}=\binom{\xi}{\eta}$. This can be written as follows:

$$\begin{cases}-\Delta[\varphi(0,0)\xi]=f(0,0)\xi,\\ -\Delta[\psi(0,0)\eta]=g(0,0)\eta, & \text{in }\Omega,\\ \kappa_1\frac{\partial\xi}{\partial n}+\tau_1\xi=0,\\ \kappa_2\frac{\partial\eta}{\partial n}+\tau_2\eta=0, & \text{on }\partial\Omega.\end{cases}$$

The above two equations are special forms of the eigenvalue problem (2.1). If $\xi > 0$ or $\eta > 0$, we have $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,0) + f(0,0)) = 0$ or $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(0,0) + g(0,0)) = 0$, which is a contradiction. Thus 1 is not an eigenvalue of L corresponding to a positive eigenfunction.

Next we calculate $\mathrm{index}_W(F,(0,0))$. Since $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,0) + f(0,0)) > 0$, we get $r(T) > 1$, where $T := (1/\varphi(0,0))(-\Delta + M)^{-1}[f(0,0) + M\varphi(0,0)]$ by Lemma 2.4(i). Then using Krein–Rutman theorem, one can see that $r(T)$ is an eigenvalue of T with a positive eigenfunction ϕ. That is, if we consider the pair $\binom{\phi}{0}$ and $\lambda = r(T) > 1$, then there is an eigenvalue greater than one with a positive eigenfunction. By Lemma 13.1 in [1], there exists a $\sigma_0 \in (0,\rho]$ such that $\mathrm{index}_W(F,P_\sigma) = 0$ for any $0 < \sigma < \sigma_0$. On the other hand, since $(0,0)$ is isolated, there exists $\delta > 0$ such that $(0,0)$ is the only fixed point of F in P_δ. If we take $\sigma < \min\{\sigma_0,\delta\}$, then $\mathrm{index}_W(F,(0,0)) = \mathrm{index}_W(F,P_\sigma) = 0$. $\square$

Lemma 4.3. *If* $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,v_0) + f(0,v_0)) > 0$ *and* $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(u_0,0) + g(u_0,0)) > 0$, *then* $\mathrm{index}_W(F,(u_0,0)) = \mathrm{index}_W(F,(0,v_0)) = 0$.

Proof. We only calculate the index for the point $y = (u_0,0)$ since the calculation of $\mathrm{index}_W(F,(0,v_0))$ can be made similarly.

For the point $y = (u_0,0)$, observe $\overline{W}_y = C_{\kappa_1,\tau_1}(\overline{\Omega}) \oplus K_2$ and then set an operator $L := F'(u_0,0)$. By the calculation, we have

$$L = \begin{pmatrix} A(u_0,0) & \varphi_v(u_0,0)u_0 \\ 0 & \psi(u_0,0) \end{pmatrix}^{-1} (-\Delta + M)^{-1} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix},$$

where

$$\begin{cases} A(u_0,0) = \varphi_u(u_0,0)u_0 + \varphi(u_0,0), \\ \alpha = f(u_0,0) + M\varphi(u_0,0) + u_0(f_u(u_0,0) + M\varphi_u(u_0,0)), \\ \beta = u_0(f_v(u_0,0) + M\varphi_v(u_0,0)), \\ \gamma = g(u_0,0) + M\psi(u_0,0). \end{cases} \tag{4.1}$$

For simplicity, we use expressions (4.1) throughout this section.

First we prove that $I - L$ is invertible on $\overline{W}_y$. Suppose there are functions $(\xi,\eta) \in \overline{W}_y$ such that $(I - L)\binom{\xi}{\eta} = \binom{0}{0}$. Then we have

$$\begin{cases} (-\Delta + M)^{-1}(\alpha\xi + \beta\eta) = A(u_0,0)\xi + \varphi_v(u_0,0)u_0\eta, \\ (-\Delta + M)^{-1}[(g(u_0,0) + M\psi(u_0,0))\eta] = \psi(u_0,0)\eta. \end{cases} \tag{4.2}$$

The second equation in (4.2) implies

$$\begin{cases} -\Delta[\psi(u_0,0)\eta] = g(u_0,0)\eta, & \text{in } \Omega, \\ \kappa_2 \frac{\partial\eta}{\partial n} + \tau_2\eta = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where $\eta \in K_2$. If $\eta \not\equiv 0$, then we can consider η as a positive eigenfunction of $\Delta\psi(u_0,0) + g(u_0,0)I$ and so $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(u_0,0) + g(u_0,0)) = 0$, which is a contradiction to our assumption. Thus $\eta \equiv 0$. Substituting $\eta = 0$ in the first equation of (4.2), we have

$$\begin{cases} \Delta[A(u_0,0)\xi] + B(u_0,0)\xi = 0, & \text{in } \Omega, \\ \kappa_1 \frac{\partial\xi}{\partial n} + \tau_1\xi = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where $B(u_0,0) = f(u_0,0) + u_0 f_u(u_0,0)$. Equation (4.4) is a special form of an eigenvalue problem (2.1). If $\xi \not\equiv 0$, then 0 is an eigenvalue of $\Delta A(u_0,0) + B(u_0,0)I$, and so we have $\lambda_1^{(\kappa_1,\tau_1)}(\Delta A(u_0,0) + B(u_0,0)) \geqslant 0$, which derives a contradiction to Lemma 4.1(i). Thus $\xi \equiv 0$, i.e., $(\xi,\eta) = (0,0)$, and so $I - L$ is invertible on $\overline{W}_y$.

Next we show that L has *property* α on $\overline{W}_y$. Observe that $S_y = C_{\kappa_1,\tau_1}(\overline{\Omega}) \oplus \{0\}$ and $\overline{W}_y \setminus S_y = C_{\kappa_1,\tau_1}(\overline{\Omega}) \oplus \{K_2 \setminus \{0\}\}$. Since $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(u_0,0) + g(u_0,0)) > 0$ from the assumption, $r(T) > 1$, where $T := (1/\psi(u_0,0))(-\Delta + M)^{-1}[g(u_0,0) + M\psi(u_0,0)]$ by Lemma 2.4(i), and so $r(T)$ is an eigenvalue of T with a corresponding positive eigenfunction $\phi \in K_2 \setminus \{0\}$ by Krein–Rutman theorem. Set $t := 1/r(T)$. Then $t \in (0,1)$ and $(0,\phi) \in \overline{W}_y \setminus S_y$. Thus

$$
\begin{aligned}
(I - tL)\begin{pmatrix} 0 \\ \phi \end{pmatrix} &= \begin{bmatrix} -\frac{t}{A(u_0,0)}(-\Delta + M)^{-1}(\beta\phi) + \frac{t\varphi_v(u_0,0)u_0}{A(u_0,0)\psi(u_0,0)}(-\Delta + M)^{-1}(\gamma\phi) \\ \phi - \frac{t}{\psi(u_0,0)}(-\Delta + M)^{-1}(\gamma\phi) \end{bmatrix} \\
&= \begin{bmatrix} -\frac{t}{A(u_0,0)}(-\Delta + M)^{-1}(\beta\phi) + \frac{t\varphi_v(u_0,0)u_0}{A(u_0,0)\psi(u_0,0)}(-\Delta + M)^{-1}(\gamma\phi) \\ \phi - \frac{1}{r(T)}T\phi \end{bmatrix} \\
&= \begin{bmatrix} -\frac{t}{A(u_0,0)}(-\Delta + M)^{-1}(\beta\phi) + \frac{t\varphi_v(u_0,0)u_0}{A(u_0,0)\psi(u_0,0)}(-\Delta + M)^{-1}(\gamma\phi) \\ 0 \end{bmatrix}.
\end{aligned}
$$

Since $I - L$ is invertible on $\overline{W}_y$, $(I - L)\binom{0}{t\phi} \in \overline{W}_y = C_{\kappa_1,\tau_1}(\overline{\Omega}) \oplus K_2$, and so

$$
-\frac{t}{A(u_0,0)}(-\Delta + M)^{-1}(\beta\phi) + \frac{t\varphi_v(u_0,0)u_0}{A(u_0,0)\psi(u_0,0)}(-\Delta + M)^{-1}(\gamma\phi) \in C_{\kappa_1,\tau_1}(\overline{\Omega}).
$$

This implies that $(I - tL)\binom{0}{\phi} \in S_y$, i.e., L has *property* α. Therefore $\mathrm{index}_W(F,(u_0,0)) = 0$ by Theorem 2.12(i). $\square$

Lemma 4.4. *If* $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,v_0) + f(0,v_0)) < 0$ *and* $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(u_0,0) + g(u_0,0)) < 0$, *then* $\mathrm{index}_W(F,(u_0,0)) = \mathrm{index}_W(F,(0,v_0)) = 1$.

Proof. We only calculate $\mathrm{index}_W(F,(u_0,0))$ since we can make a similar argument for $\mathrm{index}_W(F,(0,v_0))$.

Recall $\overline{W}_y = C_{\kappa_1,\tau_1}(\overline{\Omega}) \oplus K_2$, $S_y = C_{\kappa_1,\tau_1}(\overline{\Omega}) \oplus \{0\}$, $\overline{W}_y \setminus S_y = C_{\kappa_1,\tau_1}(\overline{\Omega}) \oplus \{K_2 \setminus \{0\}\}$, and let $L = F'(u_0,0)$. Assume that $\binom{\xi}{\eta}$ is an eigenfunction of L corresponding to the eigenvalue $\lambda \geqslant 1$. Then we have

$$
\begin{cases} (-\Delta + M)^{-1}(\alpha\xi + \beta\eta) = A(u_0,0)(\lambda\xi) + \varphi_v(u_0,0)u_0(\lambda\eta), \\ (-\Delta + M)^{-1}[(g(u_0,0) + M\psi(u_0,0))\eta] = \psi(u_0,0)(\lambda\eta). \end{cases} \tag{4.5}
$$

By Lemma 2.4(ii), our assumption implies $r((1/\psi(u_0,0))(-\Delta + M)^{-1}[g(u_0,0) + M\psi(u_0,0)]) < 1$, and so $\eta \equiv 0$. Substituting $\eta = 0$ in the first equation of (4.5), we can similarly derive $\xi \equiv 0$ by using Lemma 4.1(i) and again Lemma 2.4(ii). This implies that $I - L$ is invertible on $\overline{W}_y$ and L does not have an eigenvalue which is greater than or equal to one.

Now we suppose that L has *property* α on $\overline{W}_y$. Then there exists t such that $0 < t < 1$ and functions $(\phi_1, \phi_2) \in \overline{W}_y \setminus S_y$ such that $(I - tL)\binom{\phi_1}{\phi_2} \in S_y$. So we get $\phi_2 - (t/\psi(u_0,0))(-\Delta+M)^{-1}(g(u_0,0)+M\psi(u_0,0))\phi_2 = 0$. Since $\phi_2 \in K_2 \setminus \{0\}$, we may conclude $1/t > 1$ is an eigenvalue of the operator $(1/\psi(u_0,0))(-\Delta+M)^{-1}(g(u_0,0) + M\psi(u_0,0))$, which is a contradiction to Lemma 2.4(ii). This shows that L does not have *property* α on $\overline{W}_y$. Thus by Theorem 2.12(ii), we conclude that $\text{index}_W(F,(u_0,0)) = (-1)^\sigma$, where σ is the sum of the multiplicities of the eigenvalues of L which are greater than 1. Therefore we have $\text{index}_W(F,(u_0,0)) = 1$. □

Lemma 4.5. *If* $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0,v_0) + f(0,v_0)) = 0$ *and* $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(u_0,0) + g(u_0,0)) = 0$, *then* $\text{index}_W(F,(u_0,0)) = \text{index}_W(F,(0,v_0)) = 1$.

Proof. We only calculate $\text{index}_W(F,(u_0,0))$.

Define a homotopy F_μ by $F_\mu := G^{-1} \circ H_\mu$ for $\mu \in [0,1]$, where

$$H_\mu(u,v) = \bigl((-\Delta+M)^{-1}s_1(u,v), (-\Delta+M)^{-1}s_{2,\mu}(u,v)\bigr),$$
$$s_1(u,v) = u\bigl(f(u,v) + M\varphi(u,v)\bigr), \qquad s_{2,\mu}(u,v) = v\bigl(g(u,v) - \mu + M\psi(u,v)\bigr).$$

Clearly, $(u_0,0)$ is a fixed point of F_μ for all $\mu \in [0,1]$ and $F_0 = F$. Also we can easily verify that every fixed points of F_μ satisfy $u(x) \leqslant Q$ and $v(x) \leqslant R$. Hence F_μ has no fixed points on $\partial D \times [0,1]$. By the homotopy invariance property of index, $\text{index}_W(F,(u_0,0)) = \text{index}_W(F_\mu,(u_0,0))$.

Now we show that $\text{index}_W(F_\mu,(u_0,0)) = 1$. For the point $y = (u_0,0)$, $\overline{W}_y = C_{\kappa_1,\tau_1}(\overline{\Omega}) \oplus K_2$ and set an operator $L_\mu := F'_\mu(u_0,0)$. Then we have

$$L_\mu = \begin{pmatrix} A(u_0,0) & \varphi_v(u_0,0)u_0 \\ 0 & \psi(u_0,0) \end{pmatrix}^{-1} (-\Delta+M)^{-1} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma^* \end{pmatrix},$$

where $A(u_0,0), \alpha, \beta$ are defined in (4.1) and $\gamma^* = g(u_0,0) - \mu + M\psi(u_0,0)$. Fix $\mu > 0$. Suppose $\binom{\xi}{\eta}$ is an eigenfunction of L_μ corresponding to the eigenvalue $\lambda \geqslant 1$. Then η satisfies $\lambda\eta\psi(u_0,0) = (-\Delta+M)^{-1}(\gamma^*\eta)$, i.e., $\Delta[A^*(x)\eta] + B^*(x)\eta = 0$ in Ω and $\kappa_2(\partial\eta/\partial n) + \tau_2\eta = 0$ on $\partial\Omega$, where $A^*(x) = \psi(u_0,0)$ and $B^*(x) = g(u_0,0) + ((1-\lambda)/\lambda)(g(u_0,0) + M\psi(u_0,0)) - \mu/\lambda$. If $\eta \not\equiv 0$, then η is nonnegative and nonzero. This implies $\lambda_1^{(\kappa_2,\tau_2)}(\Delta A^*(x) + B^*(x)) = 0$. Since $\lambda \geqslant 1$ and $\mu > 0$, we have $0 = \lambda_1^{(\kappa_2,\tau_2)}(\Delta A^*(x) + B^*(x)) < \lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(u_0,0) + g(u_0,0))$ by Lemma 2.1(i), which is a contradiction to our assumption. Thus $\eta \equiv 0$. So ξ satisfies $\lambda A(u_0,0)\xi = (-\Delta + M)^{-1}(\alpha\xi)$, and so $\Delta[A(u_0,0)\xi] + (B(u_0,0) + (1-\lambda)/\lambda\alpha)\xi = 0$ in Ω and $\kappa_1(\partial\xi/\partial n) + \tau_1\xi = 0$ on $\partial\Omega$. If $\xi \not\equiv 0$, then 0 is an eigenvalue of $\Delta A(u_0,0) + (B(u_0,0) + ((1-\lambda)/\lambda) \times \alpha)I$, and so $\lambda_1^{(\kappa_1,\tau_1)}(\Delta A(u_0,0) + B(u_0,0) + ((1-\lambda)/\lambda)\alpha) \geqslant 0$. Since $\lambda \geqslant 1$, we get

$$\lambda_1^{(\kappa_1,\tau_1)}\bigl(\Delta A(u_0,0) + B(u_0,0)\bigr) \geqslant 0$$

by Lemma 2.1(ii), which is also a contradiction to Lemma 4.1(i). Hence L_μ has no eigenvalue greater than or equal to one. This implies that $I - L_\mu$ is invertible on $\overline{W}_y$ and $r(L_\mu) < 1$. As in Lemma 4.4, one can easily check that L_μ does not have *property* α on $\overline{W}_y$. Therefore we can conclude that $\text{index}_W(F_\mu,(u_0,0)) = 1$ by Theorem 2.12(ii). □

Lemma 4.6. *For an open set D' in W*, $\mathrm{index}_W(F, D') = 1$.

Proof. Clearly, ∂D contains no fixed points of F. Thus $\mathrm{index}_W(F, D')$ is well-defined. Define an operator F_μ by $F_\mu = G^{-1} \circ H_\mu$ for $\mu \in [0, 1]$, where

$$H_\mu(u, v) = \big((-\Delta + M)^{-1} s_{1,\mu}(u, v), (-\Delta + M)^{-1} s_{2,\mu}(u, v)\big),$$
$$s_{1,\mu} = u\big(\mu f(u, v) + M\varphi(u, v)\big), \qquad s_{2,\mu} = v\big(\mu g(u, v) + M\psi(u, v)\big).$$

Then clearly $F = F_1$ and, for each μ, a fixed point of F_μ is a solution of the problem

$$\begin{cases} -\Delta[\varphi(u, v)u] = \mu u f(u, v), \\ -\Delta[\psi(u, v)v] = \mu v g(u, v), & \text{in } \Omega, \\ \kappa_1 \frac{\partial u}{\partial n} + \tau_1 u = 0, \\ \kappa_2 \frac{\partial v}{\partial n} + \tau_2 v = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Note that the converse is also true. As in Lemma 3.3, we can see that every fixed points of F_μ satisfy $u(x) \leqslant Q$ and $v(x) \leqslant R$ in $\overline{\Omega}$ for each $\mu \in [0, 1]$, and so every fixed points of F_μ are in D but not on ∂D. Thus the homotopy invariance property of index shows that $\mathrm{index}_W(F_\mu, D')$ is independent of μ. So $\mathrm{index}_W(F, D') = \mathrm{index}_W(F_1, D') = \mathrm{index}_W(F_0, D')$. Noting that if $\mu = 0$, then (4.6) has only the trivial solution $(0, 0)$, we get $\mathrm{index}_W(F_0, D') = \mathrm{index}_W(F_0, (0, 0))$.

For the point $y = (0, 0)$, observe that $\overline{W}_y = K_1 \oplus K_2$, $S_y = \{0\} \oplus \{0\}$, and $\overline{W}_y \setminus S_y = (K_1 \oplus K_2) \setminus \{(0, 0)\}$. Set $L := F_0'(0, 0)$, then it is easy to check that $r(L) < 1$. This implies that $I - L$ is invertible on $\overline{W}_y$ and L does not have *property* α on $\overline{W}_y$, and so we may conclude $\mathrm{index}_W(F_0, (0, 0)) = 1$ by Theorem 2.12(ii). □

Now using Lemmas 4.2–4.6, we give the proof of Theorem 3.4.

Proof of Theorem 3.4. We show that if one of the conditions is satisfied in Theorem 3.4, then F has a positive fixed point in D'. By Lemma 3.3, $(0, 0), (u_0, 0), (0, v_0) \in D'$. Suppose that F has no positive fixed point in D'. Then by Lemma 4.6 and the additivity of index, we have

$$\begin{aligned} &\mathrm{index}_W\big(F, (0, 0)\big) + \mathrm{index}_W\big(F, (u_0, 0)\big) + \mathrm{index}_W\big(F, (0, v_0)\big) \\ &\quad = \mathrm{index}_W(F, D') = 1. \end{aligned} \tag{4.7}$$

If $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0, v_0) + f(0, v_0)) > 0$ and $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(u_0, 0) + g(u_0, 0)) > 0$, then by Lemmas 4.2 and 4.3,

$$\mathrm{index}_W\big(F, (0, 0)\big) + \mathrm{index}_W\big(F, (u_0, 0)\big) + \mathrm{index}_W\big(F, \big(0, v_0\big)\big) = 0,$$

which is a contradiction to (4.7). By the similar argument, if $\lambda_1^{(\kappa_1,\tau_1)}(\Delta\varphi(0, v_0) + f(0, v_0))$ and $\lambda_1^{(\kappa_2,\tau_2)}(\Delta\psi(u_0, 0) + g(u_0, 0))$ are both negative or both equal to zero, then we can also derive contradiction by using Lemma 4.2, 4.4 and 4.5. Therefore system (1.1) must have a positive solution in D'. □

5. Examples

In this section, we give some remarks and consider some special cases for model (1.1).

Remark 5.1. Hypothesis (H3.2) has been used only in the calculation of Jacobian of the map $G(u,v) = (\varphi(u,v)u, \psi(u,v)v)$ to show the invertibility of $G(u,v)$. So the same results of Theorem 3.4 can be obtained if (H3.2) is replaced by

(H3.2*) The map $G(u,v) = (\varphi(u,v)u, \psi(u,v)v)$ has the continuous inverse.

Consider the following competitive interacting systems between two species with nonlinear self-cross diffusions:

$$\begin{cases} -\Delta[(\alpha_1 + \beta_{11}u + \beta_{12}v)^m u] = (a_1 - b_{11}u^k - b_{12}v)u, \\ -\Delta[(\alpha_2 + \beta_{21}u + \beta_{22}v)^n v] = (a_2 - b_{21}u - b_{22}v^l)v, \quad \text{in } \Omega, \\ \kappa_1 \frac{\partial u}{\partial n} + \tau_1 u = 0, \\ \kappa_2 \frac{\partial v}{\partial n} + \tau_2 v = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\alpha_i, a_i, \beta_{ij}, b_{ij}$ are nonnegative constants with $\alpha_i > 0$, $b_{ii} > 0$, $\kappa_i^2 + \tau_i^2 \neq 0$ for $i, j = 1, 2$, and $m, n, k, l > 0$.

Lemma 5.2. *Assume that $mn \leqslant 1$ or $\beta_{11}\beta_{22} - \beta_{12}\beta_{21} \geqslant 0$ in* (5.1). *Then* (5.1) *has a positive solution if $a_1 > \alpha_1^m \lambda_1^{(\kappa_1,\tau_1)}(-\Delta)$, $a_2 > \alpha_2^n \lambda_1^{(\kappa_2,\tau_2)}(-\Delta)$ and one of the following conditions holds*:

(a) $a_1 > \lambda_1^{(\kappa_1,\tau_1)}(-\Delta(\alpha_1 + \beta_{12}v_0)^m + b_{12}v_0)$, $a_2 > \lambda_1^{(\kappa_2,\tau_2)}(-\Delta(\alpha_2 + \beta_{21}u_0)^n + b_{21}u_0)$;
(b) $a_1 < \lambda_1^{(\kappa_1,\tau_1)}(-\Delta(\alpha_1 + \beta_{12}v_0)^m + b_{12}v_0)$, $a_2 < \lambda_1^{(\kappa_2,\tau_2)}(-\Delta(\alpha_2 + \beta_{21}u_0)^n + b_{21}u_0)$;
(c) $a_1 = \lambda_1^{(\kappa_1,\tau_1)}(-\Delta(\alpha_1 + \beta_{12}v_0)^m + b_{12}v_0)$, $a_2 = \lambda_1^{(\kappa_2,\tau_2)}(\Delta(\alpha_2 + \beta_{21}u_0)^n + b_{21}u_0)$.

Proof. Comparing system (5.1) with (1.1), we note that $\varphi(u,v) = (\alpha_1 + \beta_{11}u + \beta_{12}v)^m$, $\psi(u,v) = (\alpha_2 + \beta_{21}u + \beta_{22}v)^n$, $f(u,v) = a_1 - b_{11}u^k - b_{12}v$, and $g(u,v) = a_2 - b_{21}u - b_{22}v^l$. One can easily check that system (5.1) satisfies hypotheses (H3.1), (H3.3)–(H3.5). One can show that hypothesis (H3.2*) is also satisfied by the simple calculation, in fact,

$$\begin{aligned} \frac{\partial G(u,v)}{\partial(u,v)} &= \frac{\partial}{\partial(u,v)}\big(\varphi(u,v)u, \psi(u,v)v\big) \\ &> (\alpha_1 + \beta_{11}u + \beta_{12}v)^{m-1}(\alpha_2 + \beta_{21}u + \beta_{22}v)^{n-1} \\ &\quad \times \big\{\big(\beta_{12}\beta_{21}(1 - mn) + \beta_{11}\beta_{22}mn\big)uv\big\} \geqslant 0. \end{aligned}$$

The last inequality follows from the assumption $mn \leqslant 1$ or $\beta_{11}\beta_{22} - \beta_{12}\beta_{21} \geqslant 0$. The assumptions $a_1 > \alpha_1^m \lambda_1^{(\kappa_1,\tau_1)}(-\Delta)$ and $a_2 > \alpha_2^n \lambda_1^{(\kappa_2,\tau_2)}(-\Delta)$ ensure the existence of $u_0 > 0$ and $v_0 > 0$. So if one of the given conditions is satisfied, then we may conclude that system (5.1) has a positive solution by Theorem 3.4. □

Corollary 5.3. *Consider system* (5.1) *again with* $a_1 > \alpha_1^m \lambda_1^{(\kappa_1,\tau_1)}(-\Delta)$ *and* $a_2 > \alpha_2^n \lambda_1^{(\kappa_2,\tau_2)}(-\Delta)$. *Then system* (5.1) *has a positive solution either if*

(i) *the cross diffusion pressures* β_{12} *and* β_{21} *are sufficiently large for fixed* $\alpha_i, a_i, \beta_{ii}, b_{ij}$ *for* $i, j = 1, 2$, *or*
(ii) *the coefficients of inter-specific competitions* b_{12}, b_{21} *are sufficiently large for fixed* $\alpha_i, a_i, \beta_{ij}, b_{ii}$ *for* $i, j = 1, 2$.

Proof. (i) For the semitrivial solutions u_0 and v_0 of (5.1), one can have $\partial u_0/\partial n \leqslant 0$ and $\partial v_0/\partial n \leqslant 0$ on $\partial\Omega$. In fact, if $\kappa_1 \neq 0$, then $\partial u_0/\partial n = (-\tau_1/\kappa_1)u_0 \leqslant 0$ on $\partial\Omega$. When $\kappa_1 = 0$, $\partial u_0/\partial n \leqslant 0$ on $\partial\Omega$ since $u > 0$ in Ω and $u = 0$ on $\partial\Omega$. Also one can easily see $\partial v_0/\partial n \leqslant 0$ on $\partial\Omega$ by the same reason. Using these facts, one can derive $(\partial/\partial n)((\alpha_1 + \beta_{12}v_0)^m) \leqslant 0$ and $(\partial/\partial n)((\alpha_2 + \beta_{21}u_0)^n) \leqslant 0$ on $\partial\Omega$ by the simple calculation, and so there exist constants $\beta_{12}^* > 0$, $\beta_{21}^* > 0$ such that $\lambda_1^{(\kappa_1,\tau_1)}(\Delta(\alpha_1 + \beta_{12}v_0)^m + a_1 - b_{12}v_0) < 0$ and $\lambda_1^{(\kappa_2,\tau_2)}(\Delta(\alpha_2 + \beta_{21}u_0)^n + a_2 - b_{21}u_0) < 0$ for all $\beta_{12} > \beta_{12}^*$, $\beta_{21} > \beta_{21}^*$ by Remark 2.2. Thus the result follows from Theorem 3.4. Also (ii) can be shown similarly. □

Remark 5.4. In [26], the author showed that system (5.1) has a positive solution if condition (a) or (b) is satisfied in Lemma 5.2 when $m, n, k, l \equiv 1$ and $\kappa_i = 0$ for $i = 1, 2$, i.e., when the diffusions and the growth rates are linear under homogeneous Dirichlet boundary conditions. Furthermore, we can see that, using our existence theorem, system (5.1) has also a positive solution if condition (c) is satisfied.

In the following corollary, assume $m, n, k, l = 1$. System (5.1) is called *mild competition* if $b_{11}b_{22} - b_{12}b_{21} > 0$. A pair of numbers $(\hat{u}, \hat{v})$ is said to be an *equilibrium point* if the growth rates $f(\hat{u}, \hat{v}) = g(\hat{u}, \hat{v}) = 0$ for $x \in \Omega$. A domain Ω is termed *large* if it contains a ball of large radius.

Corollary 5.5. *Consider system* (5.1) *with* $m, n, k, l = 1$. *Assume* $a_i > \alpha_i \lambda_1^{(\kappa_i,\tau_i)}(-\Delta)$ *for* $i = 1, 2$.

(i) *If*

$$\lambda_1^{(\kappa_1,\tau_1)}(-\Delta) < \frac{a_1 b_{22} - a_2 b_{12}}{\alpha_1 b_{22} + a_2 \beta_{12}} \quad \text{and} \quad \lambda_1^{(\kappa_2,\tau_2)}(-\Delta) < \frac{a_2 b_{11} - a_1 b_{21}}{\alpha_2 b_{11} + a_1 \beta_{21}},$$

then (5.1) *has a positive solution.*
(ii) *If a mild competition* (5.1) *has a positive equilibrium point when the domain* Ω *is large, then it has a positive solution.*

Proof. (i) The assumptions $a_i > \alpha_i \lambda_1^{(\kappa_i,\tau_i)}(-\Delta)$, for $i = 1, 2$, guarantee the existence of the semitrivial solutions $u_0 > 0$ and $v_0 > 0$. We can see that these semitrivial solutions satisfy $u_0 \leqslant a_1/b_{11}$ and $v_0 \leqslant a_2/b_{22}$ by Lemma 2.8. From the assumptions, we have $\lambda_1^{(\kappa_1,\tau_1)}(\Delta(\alpha_1+\beta_{12}(a_2/b_{22}))+a_1-b_{12}(a_2/b_{22}))>0$ and $\lambda_1^{(\kappa_2,\tau_2)}(\Delta(\alpha_2+\beta_{21}(a_1/b_{11}))+a_2-b_{21}(a_1/b_{11}))>0$. Note that

$$\frac{f(0,v)}{\varphi(0,v)}=\frac{a_1-b_{12}v}{\alpha_1+\beta_{12}v},\qquad \frac{g(u,0)}{\psi(u,0)}=\frac{a_2-b_{21}u}{\alpha_2+\beta_{21}u}$$

are monotone decreasing in $v\geqslant 0$ and $u\geqslant 0$, respectively. Using these facts and Lemma 2.1(ii), we can derive $\lambda_1^{(\kappa_1,\tau_1)}(\Delta(\alpha_1+\beta_{12}v_0)+a_1-b_{12}v_0)>0$ and $\lambda_1^{(\kappa_1,\tau_1)}(\Delta(\alpha_2+\beta_{21}u_0)+a_2-b_{21}u_0)>0$, and so (5.1) has a positive solution by Theorem 3.4.

(ii) The equilibrium point is

$$(\hat{u},\hat{v})=\left(\frac{a_1b_{22}-a_2b_{12}}{b_{11}b_{22}-b_{21}b_{12}},\frac{a_2b_{11}-a_1b_{21}}{b_{11}b_{22}-b_{21}b_{12}}\right).$$

So the result follows from the fact that $\lambda_1^{(\kappa_i,\tau_i)}(-\Delta)$ is small when Ω is large. □

Remark 5.6. (i) In the self-diffusions case, we do not have to assume (H3.2) because the map $G(u,v):=(\varphi(u)u,\psi(v)v)$ has always the continuous inverse if we assume (H3.1).

(ii) One can see that an a priori bound of positive solutions to the general model (1.1) with self-diffusion rates (i.e., $\varphi:=\varphi(u)$, $\psi:=\psi(v)$) is affected by the growth rates only by Lemma 3.3. (In fact, $u(x)\leqslant G_{\varphi_0}^{-1}(\varphi(C_1)C_1)=C_1$ and $v(x)\leqslant G_{\psi_0}^{-1}(\psi(C_2)C_2)=C_2$.) Ultimately, we do not need assumption (H3.5) in the self-diffusions model since (H3.5) was used only in the calculation of an a priori bound of the system with cross diffusion rates.

Consider the following systems of competing interactions with self-diffusion rates:

$$\begin{cases}-\Delta[(\alpha_1+\beta_1u)^m u]=(a_1-b_{11}u^k-b_{12}v)u,\\ -\Delta[(\alpha_2+\beta_2v)^n v]=(a_2-b_{21}u-b_{22}v^l)v, \quad \text{in } \Omega,\\ \kappa_1\frac{\partial u}{\partial n}+\tau_1u=0,\\ \kappa_2\frac{\partial v}{\partial n}+\tau_2v=0, \quad \text{on } \partial\Omega,\end{cases}\tag{5.2}$$

where $\alpha_i,a_i,\beta_i,b_{ij}$ are nonnegative constants with $\alpha_i>0$, $b_{ii}>0$, $\kappa_i^2+\tau_i^2\neq 0$ for $i=1,2$, and $m,n,k,l>0$.

Corollary 5.7. *Suppose $a_1>\alpha_1^m\lambda_1^{(\kappa_1,\tau_1)}(-\Delta)$ and $a_2>\alpha_2^n\lambda_1^{(\kappa_2,\tau_2)}(-\Delta)$. If*

$$\lambda_1^{(\kappa_1,\tau_1)}(-\Delta)<\frac{a_1b_{22}^{1/l}-a_2^{1/l}b_{12}}{\alpha_1^m b_{22}^{1/l}}\quad \textit{and}\quad \lambda_1^{(\kappa_2,\tau_2)}(-\Delta)<\frac{a_2b_{11}^{1/k}-a_1^{1/k}b_{21}}{\alpha_2^n b_{11}^{1/k}},$$

then (5.2) *has a positive solution.*

Proof. Observing $(a_1-b_{12}v)/\alpha_1^m$ and $(a_2-b_{21}u)/\alpha_2^n$ are monotone decreasing in $v\geqslant 0$ and $u\geqslant 0$, respectively, we can get the desired result by the similar manner as in Corollary 5.5(i). □

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